Lecture 12 :Rates of Change in The Natural and Social Sciences

When $y = f(x)$, $\frac{dy}{dx}$ denotes the rate of change of the function $f(x)$ with respect to x.
Recall the average rate of change of y with respect to x in the interval $[x_1, x_2]$ is

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{\Delta y}{\Delta x}.$$

The instantaneous rate of change of the function $y = f(x)$ is denoted by $\frac{dy}{dx}$.
The instantaneous rate of change of the function $y = f(x)$ when $x = a$ is denoted by

$$\left.\frac{dy}{dx}\right|_{x=a} \quad \text{or} \quad f'(a).$$

In this section we see some common uses of the rate of change in the sciences.

The units used to measure rate of change are units used for y per unit of x.

1. Physics: Objects Moving in a straight Line, Velocity and Acceleration.

Average and Instantaneous Velocity

If an object moves along a line with position $s = f(t)$.

The **average velocity** of the object over the time interval $[a, a + \Delta t]$ is the slope of the secant line between $(a, f(a))$ and $(a + \Delta t, f(a + \Delta t))$;

$$\frac{\Delta s}{\Delta t} = \frac{f(a + \Delta t) - f(a)}{\Delta t}.$$

The **instantaneous velocity** of the object at a is the slope of the tangent line to the position function at $(a, f(a))$;

$$v(a) = \left.\frac{ds}{dt}\right|_{t=a} = \lim_{\Delta t \to 0} \frac{f(a + \Delta t) - f(a)}{\Delta t} = f'(a).$$

Velocity, Speed and Acceleration.

$$\text{velocity at time } t: \qquad v = \frac{ds}{dt} = f'(t).$$

$$\text{speed at time } t: \qquad |v| = |f'(t)|.$$

$$\text{acceleration at time } t: \qquad a = \frac{dv}{dt} = \frac{d^2 s}{dt^2} = f''(t).$$

Example The position of a particle moving along a horizontal straight line is given by the equation:

$$s(t) = t^3 - 9t^2 + 24t.$$

where $s(t)$ is measured in feet and t in seconds.

(a) Find the velocity at time t.

(b) What is the velocity after 3 seconds?

(c) When is the particle at rest?

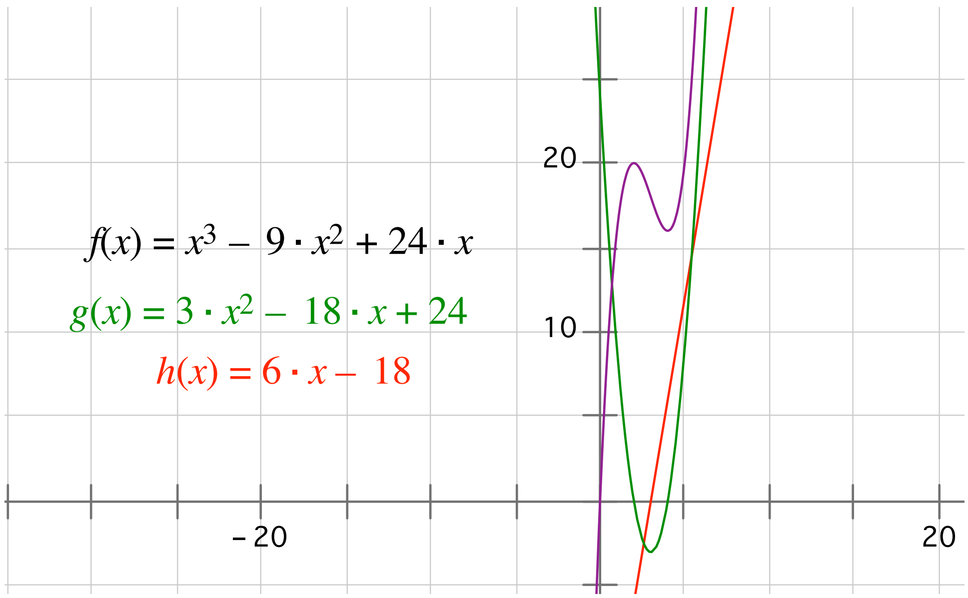
(d) When is the particle moving forwards?

(e) Draw a diagram to represent the motion of the particle.

(f) Find the distance travelled by the particle during the first three seconds.

(g) Find the acceleration at time $t = 4$ seconds.

(h) The following graph shows the position, velocity and acceleration functions for $0 \leq t \leq 10$ seconds. (where x is substituted for t).



Use the graph to help identify where the particle is moving forwards/backwards, speeding up and slowing down.

Note The particle is **speeding up** when velocity and acceleration have the same sign, the particle is **slowing down** when velocity and acceleration have opposite signs.

Example: Motion in a Gravitational field. Suppose a stone is thrown vertically upwards with an initial velocity of 48 ft/s from a bridge 160 ft above a river. By Newton's laws of motion, the position of the stone (measured as the height above the river) after t seconds is

$$s(t) = -16t^2 + 48t + 160,$$

where $s = 0$ is the level of the river.

(a) Find the velocity and acceleration functions.

(b) What is the highest point above the river reached by the stone?

(c) With what velocity will the stone strike the river?

Current

A change in electrical charge involves a flow of electrons creating a current. If $Q(t)$ is the quantity of charge (measured in Coulombs (C)) that has passed through a point in a wire up to time t , the rate of change of the function $Q(t)$ with respect to time is the current at that point $I(t)$ at time t measured in coulombs per second or amperes.

Example The quantity of charge that has passed through a point in a wire up to time t(seconds) is given by

$$Q(t) = t^3 - t^2 + t + 1 \quad \text{(Coulombs)}$$

(a) Find the current at that point when $t = 2$

Current at time $t = Q'(t) = 3t^2 - 2t + 1 = I(t)$.
When $t = 2$, the current is $Q'(2) = 12 - 4 + 1 = 9$.

(b) When is the current lowest?

The current is lowest when the function $Q'(t) = 3t^2 - 2t + 1 = 3(t - \frac{1}{3})^2 + \frac{8}{9}$ is at a minimum. Since the graph of $y = Q'(t)$ is a parabola, with a minimum value at $t = \frac{1}{3}$, we have the current is at a minimum at this time.

Rate of Growth

There is much interest in the rate of growth of populations, prices, revenue etc.. in various disciplines. If $p = f(t)$ measures the quantity at time t, then the average growth rate in the interval $[a, a + \Delta t]$ is

$$\frac{\Delta p}{\Delta t} = \frac{f(a + \Delta t) = f(a)}{\Delta t}.$$

The function

$$\frac{dp}{dt} = f'(t)$$

measures the instantaneous growth rate at time t.

Example Data collected on internet usage from 1995 onwards shows that a good model for the number of Worldwide Internet users at time t is given by

$$p(t) = 3.0t^2 + 70.8t - 48.8$$

million users, where t is measures years after 1995.

Obviously this is not a perfect fit to the data! Why?

Because $P(0) = -48.8$ million.

(a) What was the average growth rate of Internet users from 2000 to 2005?

$\frac{P(10)-P(5)}{5} = \frac{959.2-380.2}{5} = 115.8$ million users per year.

(b) What was the instantaneous growth rate of the internet in 2009?

$t = 14$.
$P'(t) = 6t + 70.8$
When $t = 14$, $P'(14) = 6(14) + 70.8 = 154.8$ million users per year.

Marginal Cost/Revenue/Profit

Companies usually keep track of their costs, revenue and profit. The cost function, $C(x)$, gives the cost to produce the first x units in the manufacturing process. How this function changes with respect to x is obviously of great interest. The **Average cost** of producing x units is

$$\frac{C(x)}{x}$$

and the **marginal cost** is given by

$$C'(x).$$

The marginal cost is roughly the cost of producing an extra unit after producing x units.

Similar interpretations of the derivative apply for the revenue function, $R(x)$, and the profit function, $P(x) = R(x) - C(x)$.

Example Suppose the cost of producing x items is given by the function

$$C(x) = -0.02x^2 + 50x + 100, \quad \text{for} \quad 0 \le x \le 1000.$$

(a) What is the average cost of producing x items for $0 \le x \le 1000$?

The average cost of producing x units, where $0 \le x \le 1000$ is

$$\frac{C(x)}{x} = \frac{-0.02x^2 + 50x + 100}{x}.$$

(b) What is the marginal cost of producing x items for $0 \le x \le 1000$?
(Marginal cost as a function of x).

The marginal cost of producing x units, where $0 \le x \le 1000$ is

$$C'(x) = -0.04x + 50.$$

(c) What is the marginal cost when $x = 100$ and when $x = 900$?

The marginal cost when $x = 100$, is

$$C'(100) = -0.04(100) + 50 = 50 - 4 = 46.$$

This can be interpreted as the cost of producing an extra unit when production level is at $x = 100$.

The marginal cost when $x = 900$, is

$$C'(900) = -0.04(900) + 50 = 14.$$

This can be interpreted as the cost of producing an extra unit when production level is at $x = 900$.